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Quenched Invariance Principles via Martingale Approximation

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Abstract

In this paper we survey the almost sure central limit theorem and its functional form (quenched) for stationary and ergodic processes. For additive functionals of a stationary and ergodic Markov chain these theorems are known under the terminology of central limit theorem and its functional form, started at a point. All these results have in common that they are obtained via a martingale approximation in the almost sure sense. We point out several applications of these results to classes of mixing sequences, shift processes, reversible Markov chains, Metropolis Hastings algorithms.

1 Introduction and general considerations

In recent years there has been an intense effort towards a better understanding of the structure and asymptotic behavior of stochastic processes. For dependent sequences there are two basic techniques: approximation with independent random variables or with martingales. Each of these methods have its own strength. On one hand the processes that can be treated by coupling with an independent sequence exhibit faster rates of convergence in various limit theorems; on the other hand the class of processes that can be treated by a martingale approximation is larger. There are plenty of processes that benefit from approximation with a

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martingale. Examples are: linear processes with martingale innovations, functions of linear processes, reversible Markov chains, normal Markov chains, various dynamical systems and the discrete Fourier transform of general stationary sequences. A martingale approximation provides important information about these structures because of their rich properties. They satisfy a broad range of inequalities, they can be embedded into Brownian motion and they satisfy various asymptotic results such as the functional conditional central limit theorem and the law of the iterated logarithm. Moreover, martingale approximation provides a simple and unified approach to asymptotic results for many dependence structures. For all these reasons, in recent years martingale approximation, "coupling with a martingale", has gained a prominent role in analyzing dependent data. This is also due to important developments by Liverani (1996), Maxwell-Woodroofe (2000), Derriennic-Lin (2001 a and b, 2003), Wu-Woodroofe (2004) and developments by Peligrad-Utev (2005), Zhao-Woodroofe (2008 a and b), Volný (2007), Peligrad-Wu (2010) among others. Many of these new results, originally designed for Markov operators, (see Kipnis-Varadhan, 1986; Derriennic-Lin, 2001 b) have made their way into limit theorems for stochastic processes.

This method has been shown to be well suited to transport from the martingale to the stationary process either the conditional central limit theorem or conditional invariance principle in probability. As a matter of fact, papers by Dedecker-Merlevède-Volný (2007), Zhao and Woodroofe (2008 b), Gordin and Peligrad (2011), point out characterizations of stochastic processes that can be approximated by martingales in quadratic mean. These results are useful to treat evolutions in "annealed" media.

In this survey we address the question of limit theorems started at a point for almost all points. These type of results are also known under the name of quenched limit theorems or almost sure conditional invariance principles. Limit theorems for stochastic processes that do not start from equilibrium is timely and motivated by recent development in evolutions in quenched random environment, random walks in random media, for instance as in Rassoul-Agha and Seppäläinen (2007). Moreover recent discoveries by Volný and Woodroofe (2010 a) show that many of the central limit theorems satisfied by classes of stochastic processes in equilibrium, fail to hold when the processes are started from a point. Special attention will be devoted to normal and reversible Markov chains and several results and open problems will be pointed out. These results are very important since reversible Markov chains have applications to statistical mechanics and to Metropolis Hastings algorithms used in Monte Carlo simulations. The method of proof of this type of limiting results are approximations with martingale in an almost sure sense.

The field of limit theorems for stochastic processes is closely related to ergodic theory and dynamical systems. All the results for stationary sequences can be translated in the language

of Markov operators.

2 Limit theorems started at a point via martingale approximation

In this section we shall use the framework of strictly stationary sequences adapted to a stationary filtrations that can be introduced in several equivalent ways, either by using a measure preserving transformation or as a functional of a Markov chain with a general state space. It is just a difference of language to present the theory in terms of stationary processes or functionals of Markov chains.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. A set $A \in \mathcal{A}$ is said to be invariant if $T(A) = A$. We denote by $\mathcal{I}$ the σ-algebra of all invariant sets. The transformation T is ergodic with respect to probability $\mathbb{P}$ if each element of $\mathcal{I}$ has measure 0 or 1. Let $\mathcal{F}_0$ be a σ-algebra of $\mathcal{A}$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$ and define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let X_0 be a $\mathcal{F}_0$-measurable, square integrable and centered random variable. Define the sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$. Let $S_n = X_1 + \cdots + X_n$. For $p \geq 1$, $\|.\|_p$ denotes the norm in $\mathbb{L}_p(\Omega, \mathcal{A}, \mathbb{P})$. In the sequel we shall denote by $\mathbb{E}_0(X) = \mathbb{E}(X|\mathcal{F}_0)$.

The conditional central limit theorem plays an essential role in probability theory and statistics. It asserts that the central limit theorem holds in probability under the measure conditioned by the past of the process. More precisely this means that for any function f which is continuous and bounded we have

$$\mathbb{E}_0(f(S_n/\sqrt{n})) \to \mathbb{E}(f(\sigma N)) \text{ in probability,} \tag{2.1}$$

where N is a standard normal variable and σ is a positive constant. Usually we shall have the interpretation $\sigma^2 = \lim_{n\to\infty}\text{var}(S_n)/n$.

This conditional form of the CLT is a stable type of convergence that makes possible the change of measure with a majorizing measure, as discussed in Billingsley (1968), Rootzén (1976), and Hall and Heyde (1980). Furthermore, if we consider the associated stochastic process

$$W_n(t) = \frac{1}{\sqrt{n}} S_{[nt]},$$

where $[x]$ denotes the integer part of x, then the conditional CLT implies the convergence of the finite dimensional distributions of $W_n(t)$ to those of $\sigma W(t)$ where $W(t)$ is the standard Brownian Motion; this constitutes an important step in establishing the functional CLT

(FCLT). Note that $W_n(t)$ belongs to the space $D[0,1]$, the set of functions on $[0,1]$ which are right continuous and have left hands limits. We endow this space with the uniform topology.

By the conditional functional central limit theorem we understand that for any function f continuous and bounded on $D[0,1]$ we have

$$\mathbb{E}_0(f(W_n)) \to \mathbb{E}(f(\sigma W)) \text{ in probability.} \tag{2.2}$$

There is a considerable amount of research concerning this problem. We mention papers by Dedecker and Merlevède (2002), Wu and Woodroofe (2004) and Zhao and Woodroofe (2008 b) among others.

The quenched versions of these theorems are obtained by replacing the convergence in probability by convergence almost sure. In other words the almost sure conditional theorem states that, on a set of probability one, for any function f which is continuous and bounded we have

$$\mathbb{E}_0(f(S_n/\sqrt{n})) \to \mathbb{E}(f(\sigma N)), \tag{2.3}$$

while by almost sure conditional functional central limit theorem we understand that, on a set of probability one, for any function f continuous and bounded on $D[0,1]$ we have

$$\mathbb{E}_0(f(W_n)) \to \mathbb{E}(f(\sigma W)). \tag{2.4}$$

We introduce now the stationary process as a functional of a Markov chain.

We assume that $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary ergodic Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a Polish space $(S, \mathcal{S})$. The marginal distribution is denoted by $\pi(A) = \mathbb{P}(\xi_0 \in A)$, $A \in \mathcal{S}$. Next, let $\mathbb{L}_2^0(\pi)$ be the set of functions h such that $||h||_{2,\pi}^2 = \int h^2 d\pi < \infty$ and $\int h d\pi = 0$. Denote by $\mathcal{F}_k$ the σ–field generated by ξ_j with $j \le k$, $X_j = h(\xi_j)$. Notice that any stationary sequence $(Y_k)_{k\in\mathbb{Z}}$ can be viewed as a function of a Markov process $\xi_k = (Y_j; j \le k)$ with the function $g(\xi_k) = Y_k$. Therefore the theory of stationary processes can be imbedded in the theory of Markov chains.

In this context by the central limit theorem started at a point (quenched) we understand the following fact: let $\mathbb{P}^x$ be the probability associated with the process started from x and let $\mathbb{E}^x$ be the corresponding expectation. Then, for $\pi-$almost every x, for every continuous and bounded function f,

$$\mathbb{E}^x(f(S_n/\sqrt{n})) \to \mathbb{E}(f(\sigma N)). \tag{2.5}$$

By the functional CLT started at a point we understand that, for $\pi-$almost every x, for every function f continuous and bounded on $D[0,1]$,

$$\mathbb{E}^x(f(W_n)) \to \mathbb{E}(f(\sigma W)). \tag{2.6}$$

where, as before W is the standard Brownian motion on $[0,1]$.

It is remarkable that a martingale with square integrable stationary and ergodic differences satisfies the quenched CLT in its functional form. For a complete and careful proof of this last fact we direct to Derriennic and Lin (2001 a, page 520). This is the reason why a fruitful approach to find classes of processes for which quenched limit theorems hold is to approximate partial sums by a martingale.

The martingale approximation as a tool in studying the asymptotic behavior of the partial sums S_n of stationary stochastic processes goes back to Gordin (1969) who proposed decomposing the original stationary sequence into a square integrable stationary martingale $M_n = \sum_{i=1}^n D_i$ adapted to $(\mathcal{F}_n)$, such that $S_n = M_n + R_n$ where R_n is a telescoping sum of random variables, with the basic property that $\sup_n ||R_n||_2 < \infty$.

For proving conditional CLT for stationary sequences, a weaker form of martingale approximation was pointed out by many authors (see for instance Merlevède-Peligrad-Utev, 2006 for a survey).

An important step forward was the result by Heyde (1974) who found sufficient conditions for the decomposition

$$S_n = M_n + R_n \text{ with } \ R_n/\sqrt{n} \to 0 \text{ in } \mathbb{L}_2. \tag{2.7}$$

Recently, papers by Dedecker-Merlevède-Volný (2007) and by Zhao-Woodroofe (2008 b) deal with necessary and sufficient conditions for martingale approximation with an error term as in (2.7).

The approximation of type (2.7) is important since it makes possible to transfer from martingale the conditional CLT defined in (2.1), where $\sigma = ||D_0||_2$.

The theory was extended recently in Gordin-Peligrad (2011) who developed necessary and sufficient conditions for a martingale decomposition with the error term satisfying

$$\max_{1\le j\le n} |S_j - M_j|/\sqrt{n} \to 0 \text{ in } \mathbb{L}_2. \tag{2.8}$$

This approximation makes possible the transport from the martingale to the stationary process the conditional functional central limit theorem stated in (2.2). These results were surveyed in Peligrad (2010).

The martingale approximation of the type (2.8) brings together many disparate examples in probability theory. For instance, it is satisfied under Hannan (1973, 1979) and Heyde (1974) projective condition.

$$\mathbb{E}(X_0|\mathcal{F}_{-\infty}) = 0 \quad \text{almost surely and} \quad \sum_{i=1}^{\infty} \|\mathbb{E}_{-i}(X_0) - \mathbb{E}_{-i-1}(X_0)\|_2 < \infty; \tag{2.9}$$

It is also satisfied for classes of mixing processes; additive functionals of Markov chains with normal or symmetric Markov operators.

A very important question is to establish quenched version of conditional CLT and conditional FCLT, i.e. the invariance principles as in (2.3) and also in (2.4) (or equivalently as in (2.5) and also in (2.6)). There are many examples of stochastic processes satisfying (2.8) for which the conditional CLT does not hold in the almost sure sense. For instance condition (2.9) is not sufficient for (2.3) as pointed out by Volný and Woodroofe (2010 a). In order to transport from the martingale to the stationary process the almost sure invariance principles the task is to investigate the approximations of types (2.7) or (2.8) with an error term well adjusted to handle this type of transport. These approximations should be of the type, for every $\varepsilon > 0$

$$\mathbb{P}_0[|S_n - M_n|/\sqrt{n} > \varepsilon] \to 0 \ a.s. \text{ or } \ \mathbb{P}_0[\max_{1\leq i\leq n} |S_i - M_i|/\sqrt{n} > \varepsilon] \to 0 \ a.s. \tag{2.10}$$

where $(M_n)_n$ is a martingale with stationary and ergodic differences and we used the notation $\mathbb{P}_0(A) = \mathbb{P}(A|\mathcal{F}_0)$. They are implied in particular by stronger approximations such that

$$|S_n - M_n|/\sqrt{n} \to 0 \ a.s. \text{ or } \ \max_{1\leq i\leq n} |S_i - M_i|/\sqrt{n} \to 0 \ a.s.$$

Approximations of these types have been considered in papers by Zhao-Woodroofe (2008 a), Cuny (2011), Merlevède- Peligrad M.- Peligrad C. (2011) among others.

In the next subsection we survey resent results and point out several classes of stochastic processes for which approximations of the type (2.10) hold.

For cases where a stationary martingale approximation does not exist or cannot be pointed out, a nonstationary martingale approximation is a powerful tool. This method was occasionally used to analyze a stochastic process. Many ideas are helpful in this situation ranging from simple projective decomposition of sums as in Gordin and Lifshitz (1981) to more sophisticated tools. One idea is to divide the variables into blocks and then to approximate the sums of variables in each block by a martingale difference, usually introducing a new parameter, the block size, and changing the filtration. This method was successfully used in the literature by Philipp-Stout (1975), Shao (1995), Merlevède-Peligrad (2006), among others. Alternatively, one can proceed as in Wu-Woodroofe (2004), who constructed a nonstationary martingale approximation for a class of stationary processes without partitioning the variables into blocks.

Recently Dedecker-Merlevède-Peligrad (2012) used a combination of blocking technique and a row-wise stationary martingale decomposition in order to enlarge the class of random variables known to satisfy the quenched invariance principles. To describe this approach,

roughly speaking, one considers an integer $m = m(n)$ large but such that $n/m \to \infty$. Then one forms the partial sums in consecutive blocks of size m, $Y_j^n = X_{m(j-1)+1} + \dots + X_{mj}$, $1 \le j \le k$, $k = [n/m]$. Finally, one considers the decomposition

$$S_n = M_n^n + R_n^n, \tag{2.11}$$

where $M_n^n = \sum_{j=1}^{n} D_j^n$, with $D_j^n = Y_j^n - E(Y_j^n|\mathcal{F}_{m(j-1)})$ a triangular array of row-wise stationary martingale differences.

2.1 Functional Central limit theorem started at a point under projective criteria.

We have commented that condition (2.9) is not sufficient for the validity of the almost sure CLT started from a point. Here is a short history of the quenched CLT under projective criteria. A result in Borodin and Ibragimov (1994, ch.4, section 8) states that if $||\mathbb{E}_0(S_n)||_2$ is bounded, then the CLT in its functional form started at a point (2.4) holds. Later, Derriennic-Lin (2001 a and b, 2003) improved on this result imposing the condition $||\mathbb{E}_0(S_n)||_2 = O(n^{1/2-\epsilon})$ with $\epsilon > 0$ (see also Rassoul-Agha and Seppäläinen, 2008). A step forward was made by Cuny (2011) who improved the condition to $||\mathbb{E}_0(S_n)||_2 = O(n^{1/2}(\log n)^{-2}(\log\log n)^{-1-\delta})$ with $\delta > 0$, by using sharp results on ergodic transforms in Gaposhkin (1996).

We shall describe now the recent progress made on the functional central limit theorem started at a point under projective criteria. We give here below three classes of stationary sequences of centered square integrable random variables for which both quenched central limit theorem and its quenched functional form given in (2.3) and (2.4) hold with $\sigma^2 = \lim_{n\to\infty} \mathrm{var}(S_n)/n$, provided the sequences are ergodic. If the sequences are not ergodic then then the results still hold but with σ^2 replaced by the random variable η described as $\eta = \lim_{n\to\infty} \mathbb{E}(S_n^2|\mathcal{I})/n$ and $\mathbb{E}(\eta) = \sigma^2$. For simplicity we shall formulate the results below only for ergodic sequences.

1. **Hannan-Heyde projective criterion**. Cuny-Peligrad (2012) (see also Volný-Woodroofe, 2010 b) showed that (2.3) holds under the condition

$$\frac{\mathbb{E}(S_n|\mathcal{F}_0)}{\sqrt{n}} \to 0 \quad \text{almost surely and} \quad \sum_{i=1}^{\infty} ||\mathbb{E}_{-i}(X_0) - \mathbb{E}_{-i-1}(X_0)||_2 < \infty. \tag{2.12}$$

The functional form of this result was established in Cuny-Volný (2013).

2. **Maxwell and Woodroofe condition**. The convergence in (2.4) holds under Maxwell-

Woodroofe (2000) condition,

$$\sum_{k=1}^{\infty} \frac{||\mathbb{E}_0(S_k)||_2}{k^{3/2}} < \infty, \tag{2.13}$$

as recently shown in Cuny-Merlevède (2012). In particular both conditions (2.12) and (2.13) and is satisfied if

$$\sum_{k=1}^{\infty} \frac{||\mathbb{E}_0(X_k)||_2}{k^{1/2}} < \infty. \tag{2.14}$$

3. **Dedecker-Rio condition.** In a recent paper Dedecker-Merlevède-Peligrad (2012) proved (2.4) under the condition

$$\sum_{k \geq 0} \|X_0 \mathbb{E}_0(X_k)\|_1 < \infty. \tag{2.15}$$

The first two results were proved using almost sure martingale approximation of type (2.10). The third one was obtained using the large block method described in (2.11).

Papers by Durieu-Volný (2008) and Durieu (2009) suggest that conditions (2.12), (2.13) and (2.15) are independent. They have different areas of applications and they lead to optimal results in all these applications. Condition (2.12) is well adjusted for linear processes. It was shown in Peligrad and Utev (2005) that the Maxwell-Woodroofe condition (2.13) is satisfied by $\rho-$mixing sequences with logarithmic rate of convergence to 0. Dedecker-Rio (2000) have shown that condition (2.15) is verified for strongly mixing processes under a certain condition combining the tail probabilities of the individual summands with the size of the mixing coefficients. For example, one needs a polynomial rate on the strong mixing coefficients when moments higher than two are available. However, the classes described by projection conditions have a much larger area of applications than mixing sequences. They can be verified by linear processes and dynamical systems that satisfy only weak mixing conditions (Dedecker-Prieur 2004 and 2005, Dedecker-Merlevède-Peligrad (2012) among others). More details about the applications are given in Section 3.

Certainly, these projective conditions can easily be formulated in the language of Markov operators by using the fact that $\mathbb{E}_0(X_k) = Q(f)(\xi_0)$. In this language $\mathbb{E}_0(S_k) = (Q + Q^2 + ... + Q^k)(f)(\xi_0)$.

2.2 Functional Central limit theorem started at a point for normal and reversible Markov chains.

In 1986 Kipnis and Varadhan proved the functional form of the central limit theorem as in (2.2) for square integrable mean zero additive functionals $f \in \mathbb{L}_2^0(\pi)$ of stationary reversible

ergodic Markov chains $(\xi_n)_{n\in\mathbb{Z}}$ with transition function $Q(\xi_0, A) = P(\xi_1 \in A|\xi_0)$ under the natural assumption $var(S_n)/n$ is convergent to a positive constant. This condition has a simple formulation in terms of spectral measure ρ_f of the function f with respect to self-adjoint operator Q associated to the reversible Markov chain, namely

$$\int_{-1}^{1} \frac{1}{1-t}\rho_f(dt) < \infty. \tag{2.16}$$

This result was established with respect to the stationary probability law of the chain. (Self-adjoint means $Q = Q^*$, where Q also denotes the operator $Qf(\xi) = \int f(x)Q(\xi, dx)$; Q^* is the adjoint operator defined by $< Qf, g >=< f, Q^*g >$, for every f and g in $\mathbb{L}_2(\pi)$).

The central limit theorem (2.1) for stationary and ergodic Markov chains with normal operator Q ($QQ^* = Q^*Q$), holds under a similar spectral assumption, as discovered by Gordin-Lifshitz (1981) (see also and Borodin-Ibragimov, 1994, ch. 4 sections 7-8). A sharp sufficient condition in this case in terms of spectral measure is

$$\int_{D} \frac{1}{|1-z|}\rho_f(dz) < \infty. \tag{2.17}$$

where D is the unit disk.

Examples of reversible Markov chains frequently appear in the study of infinite systems of particles, random walks or processes in random media. A simple example of a normal Markov chain is a random walk on a compact group. Other important example of reversible Markov chain is the extremely versatile (independent) Metropolis Hastings Algorithm which is the modern base of Monte Carlo simulations.

An important problem is to investigate the validity of the almost sure central limit theorem started at a point for stationary ergodic normal or reversible Markov chains. As a matter of fact, in their remark (1.7), Kipnis-Varadhan (1986) raised the question if their result also holds with respect to the law of the Markov chain started from x, for almost all x, as in (2.6).

Conjecture: For any square integrable mean 0 function of reversible Markov chains satisfying condition (2.16) the functional central limit theorem started from a point holds for almost all points. The same question is raised for continuous time reversible Markov chains.

The answer to this question for reversible Markov chains with continuous state space is still unknown and has generated a large amount of research. The problem of quenched CLT for normal stationary and ergodic Markov chains was considered by Derriennic-Lin (2001 a) and Cuny (2011), among others, under some reinforced assumptions on the spectral condition. Concerning normal Markov chains, Derriennic-Lin (2001 a) pointed out that the central limit theorem started at a point does not hold for almost all points under condition (2.17). Furthermore, Cuny-Peligrad (2012) proved that there is a stationary and ergodic normal Markov

chain and a function $f \in \mathbb{L}_2^0(\pi)$ such that

$$\int_D \frac{|\log(|1-z|)\log\log(|1-z|)|}{|1-z|}\rho_f(dz) < \infty$$

and such that the central limit theorem started at a point fails, for $\pi-$almost all starting points.

However the condition

$$\int_{-1}^{1} \frac{(\log^+|\log(1-t)|)^2}{1-t}\rho_f(dt) < \infty, \tag{2.18}$$

is sufficient to imply central limit theorem started at a point (2.5) for reversible Markov chains for $\pi-$almost all starting points. Note that this condition is a slight reinforcement of condition (2.17).

It is interesting to note that by Cuny (2011, Lemma 2.1), condition (2.18) is equivalent to the following projective criterion

$$\sum_n \frac{(\log\log n)^2||\mathbb{E}_0(S_n)||_2^2}{n^2} < \infty. \tag{2.19}$$

Similarly, condition (2.17) in the case where Q is symmetric, is equivalent to

$$\sum_n \frac{||\mathbb{E}_0(S_n)||_2^2}{n^2} < \infty. \tag{2.20}$$

3 Applications

Here we list several classes of stochastic processes satisfying quenched CLT and quenched invariance principles. They are applications of the results given in Section 2.

3.1 Mixing processes

In this subsection we discuss two classes of mixing sequences which are extremely relevant in the study of Markov chains, Gaussian processes and dynamical systems.

We shall introduce the following mixing coefficients: For any two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ define the strong mixing coefficient $\alpha(\mathcal{A},\mathcal{B})$:

$$\alpha(\mathcal{A},\mathcal{B}) = \sup\{|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|; A \in \mathcal{A}, B \in \mathcal{B}\}.$$

The $\rho-$mixing coefficient, known also under the name of maximal coefficient of correlation $\rho(\mathcal{A},\mathcal{B})$ is defined as:

$$\rho(\mathcal{A},\mathcal{B}) = \sup\{\mathrm{Cov}(X,Y)/\|X\|_2\|Y\|_2 \ : \ X \in \mathbb{L}_2(\mathcal{A}),\ Y \in \mathbb{L}_2(\mathcal{B})\}.$$

For the stationary sequence of random variables $(X_k)_{k\in\mathbb{Z}}$, we also define $\mathcal{F}_m^n$ the σ–field generated by X_i with indices $m \le i \le n$, $\mathcal{F}^n$ denotes the σ–field generated by X_i with indices $i \ge n$, and $\mathcal{F}_m$ denotes the σ–field generated by X_i with indices $i \le m$. The sequences of coefficients $\alpha(n)$ and $\rho(n)$ are then defined by

$$\alpha(n) = \alpha(\mathcal{F}_0, \mathcal{F}^n),\ \rho(n) = \rho(\mathcal{F}_0, \mathcal{F}^n).$$

An equivalent definition for $\rho(n)$ is

$$\rho(n) = \sup\{\|\mathbb{E}(Y|\mathcal{F}_0)\|_2/\|Y\|_2 :\ Y \in \mathbb{L}_2(\mathcal{F}^n), \mathbb{E}(Y) = 0\}. \tag{3.21}$$

Finally we say that the stationary sequence is strongly mixing if $\alpha(n) \to 0$ as $n \to \infty$, and $\rho-$mixing if $\rho(n) \to 0$ as $n \to \infty$. It should be mentioned that a $\rho-$mixing sequence is strongly mixing. Furthermore, a stationary strongly mixing sequence is ergodic. For an introduction to the theory of mixing sequences we direct the reader to the books by Bradley (2007).

In some situations weaker forms of strong and $\rho-$mixing coefficients can be useful, when $\mathcal{F}^n$ is replaced by the sigma algebra generated by only one variable, X_n, denoted by $\mathcal{F}_n^n$. We shall use the notations $\tilde{\alpha}(n) = \alpha(\mathcal{F}_0, \mathcal{F}_n^n)$ and $\tilde{\rho}(n) = \rho(\mathcal{F}_0, \mathcal{F}_n^n)$.

By verifying the conditions in Section 3, we can formulate:

Theorem 1. *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary and ergodic sequence of centered square integrable random variables. The quenched CLT and its quenched functional form as in (2.3) and (2.4) hold with $\sigma^2 = \lim_{n\to\infty} var(S_n)/n$ under one of the following three conditions:*

$$\sum_{k=1}^{\infty} \frac{\tilde{\rho}(k)}{\sqrt{k}} < \infty. \tag{3.22}$$

$$\sum_{k=1}^{\infty} \frac{\rho(k)}{k} < \infty. \tag{3.23}$$

$$\sum_{k=1}^{\infty} \int_0^{\tilde{\alpha}(k)} Q^2(u)du < \infty, \tag{3.24}$$

where Q denotes the generalized inverse of the function $t \to \mathbb{P}(|X_0| > t)$.

We mention that under condition (3.23) the condition of ergodicity is redundant. Also if (3.24) holds with $\tilde{\alpha}(k)$ replaced by $\alpha(k)$, then the sequence is again ergodic.

In order to prove this theorem under (3.22) one verifies condition (2.14) via the estimate

$$\mathbb{E}(\mathbb{E}_0(X_k))^2 = \mathbb{E}(X_k\mathbb{E}_0(X_k)) \le \tilde{\rho}(k)||X_0||_2^2,$$

which follows easily from the definition of $\tilde{\rho}$.

Condition (3.23) is used to verify condition (2.13). This was verified in the Peligrad-Utev-Wu (2007) via the inequalities

$$\|\mathbb{E}(S_{2^{r+1}}|\mathcal{F}_0)\|_2 \leq c\sum_{j=0}^{r} 2^{j/2}\rho(2^j)$$

and

$$\sum_{r=0}^{\infty} \frac{\|\mathbb{E}(S_{2^r}|\mathcal{F}_0)\|_2}{2^{r/2}} \leq c\sum_{j=0}^{\infty} \rho(2^j) < \infty. \tag{3.25}$$

Furthermore (3.25) easily implies (2.13). For more details on this computation we also direct the reader to the survey paper by Merlevède-Peligrad-Utev (2006).

To get the quenched results under condition (3.24) the condition (2.15) is verified via the following identity taken from Dedecker-Rio (2000, (6.1))

$$\mathbb{E}|X_0\mathbb{E}(X_k|\mathcal{F}_0)| = \mathrm{Cov}(|X_0|(I_{\{\mathbb{E}(X_k|\mathcal{F}_0)>0\}} - I_{\{\mathbb{E}(X_k|\mathcal{F}_0)\leq 0\}}), X_k). \tag{3.26}$$

By applying now Rio's (1993) covariance inequality we obtain

$$\mathbb{E}|X_0\mathbb{E}(X_k|\mathcal{F}_0)| \leq c\int_0^{\tilde{\alpha}(k)} Q^2(u)du.$$

It is obvious that condition (3.22) requires a polynomial rate of convergence to 0 of $\tilde{\rho}(k)$; condition (3.23) requires only a logarithmic rate for $\rho(n)$. To comment about condition (3.24) it is usually used in the following two forms:

-either the variables are almost sure bounded by a constant, and then the requirement is $\sum_{k=1}^{\infty} \tilde{\alpha}(k) < \infty$.

-the variables have finite moments of order $2+\delta$ for some $\delta > 0$, and then the condition on mixing coefficients is $\sum_{k=1}^{\infty} k^{2/\delta}\tilde{\alpha}(k) < \infty$.

3.2 Shift processes.

In this sub-section we apply condition (2.13) to linear processes which are not mixing in the sense of previous subsection. This class is known under the name of one-sided shift processes, also known under the name of Raikov sums.

Let us consider a Bernoulli shift. Let $\{\varepsilon_k; k \in \mathbb{Z}\}$ be an i.i.d. sequence of random variables with $\mathbb{P}(\varepsilon_1 = 0) = \mathbb{P}(\varepsilon_1 = 1) = 1/2$ and let

$$Y_n = \sum_{k=0}^{\infty} 2^{-k-1}\varepsilon_{n-k} \quad \textit{and} \quad X_n = g(Y_n) - \int_0^1 g(x)dx\,,$$

where $g \in \mathbb{L}_2(0,1)$, $(0,1)$ being equipped with the Lebesgue measure.

By applying Proposition 3 in Maxwell and Woodroofe (2000) for verifying condition (2.13), we see that if $g \in \mathbb{L}_2(0,1)$ satisfies

$$\int_0^1 \int_0^1 [g(x) - g(y)]^2 \frac{1}{|x-y|} (\log[\log \frac{1}{|x-y|}])^t dxdy < \infty \tag{3.27}$$

for some $t > 1$, then (2.13) is satisfied and therefore (2.3) and (2.4) hold with $\sigma^2 = \lim_{n\to\infty} \mathrm{var}(S_n)/n$. A concrete example of a map satisfying (3.27), pointed out in Merlevède-Peligrad-Utev, 2006 is

$$g(x) = \frac{1}{\sqrt{x}} \frac{1}{[1 + \log(2/x)]^4} \sin(\frac{1}{x}) \ , \ 0 < x < 1.$$

3.3 Random walks on orbits of probability preserving transformation

The following example was considered in Derriennic-Lin (2007) and also in Cuny-Peligrad (2012). Let us recall the construction.

Let τ be an invertible ergodic measure preserving transformation on $(S, \mathcal{A}, \pi)$, and denote by U, the unitary operator induced by τ on $\mathbb{L}_2(\pi)$. Given a probability $\nu = (p_k)_{k\in\mathbb{Z}}$ on $\mathbb{Z}$, we consider the Markov operator Q with invariant measure π, defined by

$$Qf = \sum_{k\in\mathbb{Z}} p_k f \circ \tau^k, \quad \text{for every } f \in \mathbb{L}_1(\pi).$$

This operator is associated to the transition probability

$$Q(x, A) = \sum_{k\in\mathbb{Z}} p_k \mathbf{1}_A(\tau^k s), \qquad s \in S, A \in \mathcal{A}.$$

We assume that ν is ergodic, i.e. the the group generated by $\{k \in \mathbb{Z} : p_k > 0\}$ is $\mathbb{Z}$. As shown by Derriennic-Lin (2007), since τ is ergodic, Q is ergodic too. We assume ν is symmetric implying that the operator Q is symmetric.

Denote by Γ the unit circle. Define the Fourier transform of ν by $\varphi(\lambda) = \sum_{k\in\mathbb{Z}} p_k \lambda^k$, for every $\lambda \in \Gamma$. Since ν is symmetric, $\varphi(\lambda) \in [-1, 1]$, and if μ_f denotes the spectral measure (on Γ) of $f \in \mathbb{L}_2(\pi)$, relative to the unitary operator U, then, the spectral measure ρ_f (on $[-1,1]$) of f, relative to the symmetric operator Q is given by

$$\int_{-1}^1 \psi(s)\rho_f(ds) = \int_\Gamma \psi(\varphi(\lambda))\mu_f(d\lambda),$$

for every positive Borel function ψ on $[-1,1]$. Condition (2.19) is verified under the assumption

$$\int_\Gamma \frac{(\log^+ |\log(1-\varphi(\lambda))|)^2}{1-\varphi(\lambda)} \mu_f(d\lambda) < \infty.$$

and therefore (2.5) holds.

When ν is centered and admits a moment of order 2 (i.e. $\sum_{k\in\mathbb{Z}} k^2 p_k < \infty$), Derriennic and Lin (2007) proved that the condition $\int_\Gamma \frac{1}{|1-\varphi(\lambda)|}\mu_f(d\lambda) < \infty$, is sufficient for (2.5).

Let $a \in \mathbb{R} - \mathbb{Q}$, and let τ be the rotation by a on $\mathbb{R}/\mathbb{Z}$. Define a measure σ on $\mathbb{R}/\mathbb{Z}$ by $\sigma = \sum_{k\in\mathbb{Z}} p_k \delta_{ka}$. For that τ, the canonical Markov chain associated to Q is the random walk on $\mathbb{R}/\mathbb{Z}$ of law σ. In this setting, if $(c_n(f))$ denotes the Fourier coefficients of a function $f \in \mathbb{L}_2(\mathbb{R}/\mathbb{Z})$, condition (2.19) reads

$$\sum_{n\in\mathbb{Z}} \frac{(\log^+ |\log(1-\varphi(\mathrm{e}^{2i\pi na}))|)^2 |c_n(f)|^2}{1-\varphi(\mathrm{e}^{2i\pi na})} < \infty.$$

3.4 CLT started from a point for a Metropolis Hastings algorithm.

In this subsection we mention a standardized example of a stationary irreducible and aperiodic Metropolis-Hastings algorithm with uniform marginal distribution. This type of Markov chain is interesting since it can easily be transformed into Markov chains with different marginal distributions. Markov chains of this type are often studied in the literature from different points of view. See, for instance Rio (2009).

Let $E = [-1, 1]$ and let υ be a symmetric atomless law on E. The transition probabilities are defined by

$$Q(x, A) = (1 - |x|)\delta_x(A) + |x|\upsilon(A),$$

where δ_x denotes the Dirac measure. Assume that $\theta = \int_E |x|^{-1}\upsilon(dx) < \infty$. Then there is a unique invariant measure

$$\pi(dx) = \theta^{-1}|x|^{-1}\upsilon\,(dx)$$

and the stationary Markov chain (γ_k) generated by $Q(x, A)$ and π is reversible and positively recurrent, therefore ergodic.

Theorem 2. *Let f be a function in $\mathbb{L}_2^0(\pi)$ satisfying $f(-x) = -f(x)$ for any $x \in E$. Assume that for some positive t, $|f| \le g$ on $[-t, t]$ where g is an even positive function on E such that g is nondecreasing on $[0, 1]$, $x^{-1}g(x)$ is nonincreasing on $[0, 1]$ and*

$$\int_{[0,1]} [x^{-1}g(x)]^2 dx < \infty. \tag{3.28}$$

Define $X_k = f(\gamma_k)$. Then (2.5) holds.

Proof. Because the chain is Harris recurrent if the annealed CLT holds, then the CLT also holds for any initial distribution (see Chen, 1999), in particular started at a point. Therefore

it is enough to verify condition (2.20). Denote, as before, by $\mathbb{E}^x$ the expected value for the process started from $x \in E$. We mention first relation (4.6) in Rio (2009). For any $n \geq 1/t$

$$|\mathbb{E}^x(S_n(g))| \leq ng(1/n) + t^{-1}|f(x)| \text{ for any } x \in [-1, 1].$$

Then

$$|\mathbb{E}^x(S_n(g))|^2 \leq 2[ng(1/n)]^2 + 2t^{-2}|f(x)|^2 \text{ for any } x \in [-1, 1],$$

and so, for any $n \geq 1/t$

$$||\mathbb{E}^x(S_n)||_{2,\pi}^2 \leq 2[ng(1/n)]^2 + 2t^{-2}||f(x)||_{2,\pi}^2.$$

Now we impose condition (2.20) involving $||\mathbb{E}^x(S_n)||_{2,\pi}^2$, and note that

$$\sum\nolimits_n \frac{[ng(1/n)]^2}{n^2} < \infty \text{ if and only if (3.28) holds.}$$

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